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RESPONSES OF BLAST LOADING BY COMPLEX TIME STEP METHOD

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Blast loading is often described by means of high order functions, and step-by-step time integration algorithms are commonly used to evaluate the numerical solutions. The time step size for the Newmark method has to be very small in order to integrate the high order loading accurately. Recently, a complex time step formulation has been proposed to construct unconditionally stable higher order accurate time step integration algorithms with controllable numerical dissipation where loading with high order variation can be tackled without difficulties. The responses at the end of a time step are obtained by linearly combining the responses at various complex sub-step locations with different weighting factors. In this paper, the complex time step method is extended to evaluate the responses within a time step. The required weighting factors anywhere within a time step can be worked out systematically. Besides, there are some locations within a time step with one order higher in accuracy. A procedure is also proposed to evaluate the modified excitation at various complex sub-step locations. To verify the complex time step method, a single-degree-of freedom system subject to blast loading described by a fourth order polynomial is considered in detail. A multi-degree-of-freedom system is also analyzed. Excellent performance over the Newmark method is noted. It is possible to evaluate the responses due to blast loading by using just one time step.

1. INTRODUCTION

Blast loading is the result of an explosion that comes in the form of a shock wave consisting of a high-pressure shock front from the centre of detonation. A typical pressure–time history for a blast wave in free air is shown in Figure 1. The shock front arrives at time t_a and reaches its peak value. The pressure then decays to the ambient value which defines the positive phase duration τ. This is followed by a negative phase duration where a reversal of the air particles results in suction. The Friedlander equation

$$p(t) = p_s\left[1 - \frac{t}{\tau}\right]\exp\left\{-\frac{bt}{\tau}\right\} \tag{1}$$

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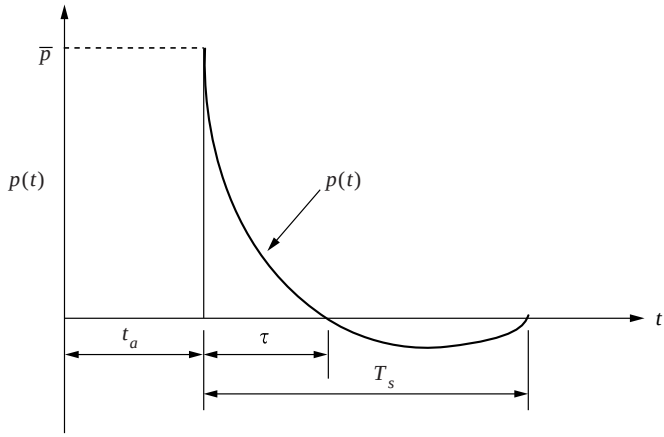

Figure 1. A typical pressure versus time curve for blast wave.

is often used to describe the decay of the pressure intensity using the exponential function [1].

In some situations, the actual blast loading is approximated by assuming linear decay for the positive phase and with the negative phase being neglected. Such approximations neglect possible effects on the response by the suction phase [2]. Bakri and Watson [3] made a comparative study on the response of a slab subject to four different types of blast loading shown in Figure 2. Each type of loading tries to approximate the actual pressure–time history obtained in an experiment. Both Lines I and II assume linear decay of the loading and ignore the negative phase. Line I uses the same peak overpressure and positive time duration as the actual loading. On the other hand, Line II uses the same peak overpressure but preserves the impulse by adjusting the positive phase duration. Line III considers only the actual positive phase of the loading and Line IV uses the full blast load in

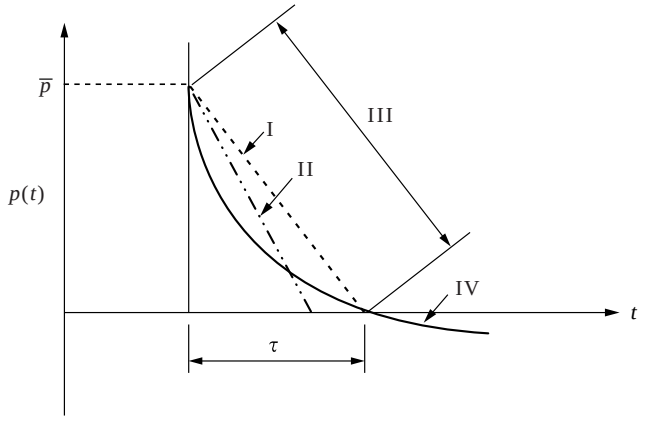

Figure 2. Idealisation of pressure–time profile.

both the positive and negative phases. The results showed that Line I over predicted the slab response while Lines II and III, both having the same impulse, gave similar responses lower than that of Line I. Line IV gave the best prediction for the deflection–time history. These observations suggest that the negative phase of blast pressure is important in order to predict accurately the responses of blast loading on structures.

Overestimating the loading, such as considering only the positive phase of the blast load, may be appropriate for defensive or design purposes since uncertainties are already present in the loading parameters. This, however, may not be the case for offensive purposes such as military target-analysis [4]. In view of the need to obtain accurate prediction of response–time history, it is of much interest to seek solution techniques suitable for blast loading

Singhal and Larson [5] used a fourth order polynomial to describe the blast wave and calculated the dynamic reduction factors of flexible panels. Both the positive and negative phases of the blast load were considered and an analytical closed form solution was obtained by Duhamel integrals. The results were then compared with different time step integration schemes. In this paper, the complex time step method is used to evaluate the responses. It is found that the present method is particularly suitable for systems subject to loading with higher order variation, such as blast loading. It is possible to evaluate the responses accurately by using just one large time step.

1.1. ALGORITHMS FOR TIME-STEP INTEGRATION

In the analysis of structural response to dynamic loading (such as blast loading), the structure is commonly modelled using the finite element method. The resulting equations are then solved by time step integration methods to obtain numerical solutions at discrete time points [6]. For a multi-degree-of-freedom system, the equations of motion after spatial discretization using the finite element method can be written as

$$[\mathbf{M}]\{\ddot{\mathbf{u}}(t)\} + [\mathbf{C}]\{\dot{\mathbf{u}}(t)\} + [\mathbf{K}]\{\mathbf{u}(t)\} = \{\mathbf{F}(t)\} \tag{2}$$

where [**M**], [**C**] and [**K**] are the mass, damping and stiffness matrices respectively, $\{\mathbf{F}(t)\}$ is the applied load vector, $\{\mathbf{u}(t)\}$ is the unknown displacement vector and dots denote differentiation with respect to time t. The initial conditions at $t=0$ are $\{\mathbf{u}(0)\}=\{\mathbf{u}_0\}$, $\{\dot{\mathbf{u}}(0)\}=\{\mathbf{v}_0\}$.

To solve equation (2) numerically using time step integration algorithms, it is desirable for the algorithms (i) to possess numerical dissipation so as to damp out the spurious high-frequency responses and (ii) to be unconditionally stable so that time steps of any size can be used without introducing numerical instability. The commonly used algorithms are the linear mutli-step algorithms such as the central difference method, Trapezoidal rule, Newmark method, Wilson-θ method, HHT-α method, Houbolt method, Park method, WBZ-α, method, Bossak method and Bazzi–Anderheggen method [7–9]. These algorithms are unconditionally stable but only second-order accurate. Third and

higher order accurate linear multi-step algorithms give more accurate numerical results but they are conditionally stable only.

Fast and accurate algorithms are useful for dynamical design, analysis and control of mechanical and structural systems. Higher-order accurate algorithms give very accurate numerical results and are good for long-term prediction of system responses and preservation of system invariant (such as energy and momentum). With the higher order algorithms, larger steps in time marching can be taken without compromising accuracy. As an alternative to the h-type refinement (decreasing time step), the higher-order algorithms can be regarded as the p-type refinement (increasing the order of approximating polynomials). Peters and Izadpanah [10] pointed out that p-version finite elements in time can be made competitive with conventional time-marching algorithms, particularly if high accuracy is needed. More discussions on the higher order accurate time step integration algorithms can be found in references [11, 12].

Nowadays, the Newmark method is still very commonly used. However, it is non-dissipative when second order accurate. Recently, Fung [13–15] proposed the complex time step method based on the Newmark method to construct unconditionally stable higher order accurate time step integration algorithms with controllable numerical dissipation. The complex time step sub-stepping procedure in Figure 3(a) is different from the linear multi-step procedure in Figure 3(b). In the complex time step method, the numerical results at different sub-step locations are evaluated independently and then combined linearly to give higher order accurate results at the end of a time step. The sub-step locations may be complex. The order of accuracy of an algorithm determines the number of sub-steps in order to advance one time step. In general, $(2n-1)$-th order accurate algorithms can be obtained by using n sub-steps. Furthermore, independent evaluation of the results at each sub-step location also enable the algorithm to be implemented on parallel computers easily so as to speed up the computation time.

There are three types of algorithmic parameters in the complex time step method: the sub-step locations β_k, the weighting factors α_k for combining the results at the sub-step locations and the desirable ultimate spectral radius μ. Here the ultimate spectral radius controls the stability property and the

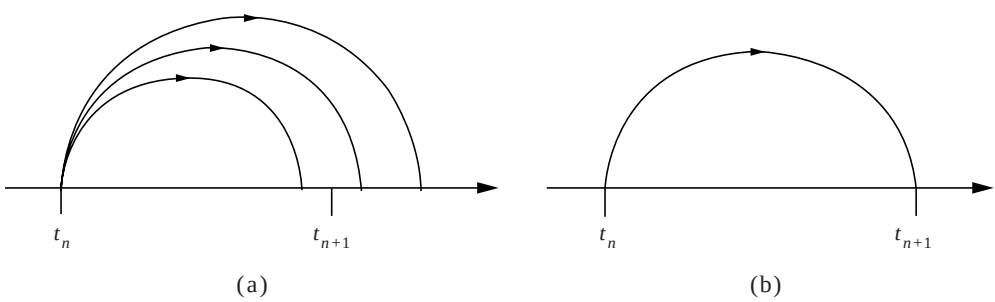

Figure 3. Time-stepping procedures: (a) complex time step method, (b) Newmark method.

numerical dissipation. It has been shown that if $-1 < \mu \leqslant 1$, the resultant algorithms are unconditionally stable for all n [14, 15]. The algorithms with $-1 < \mu < 0$ are seldom used since they are not as accurate as those with $0 \leqslant \mu \leqslant 1$. By varying μ, the algorithms can be damped in a controllable way. A special case is when $\mu = 1$, the algorithms for various n are non-dissipative with an order of accuracy increased from $2n-1$ to $2n$. It has also been shown that in order to maintain higher order accurate solutions, the excitation may need some modifications.

Since the complex time step algorithms are higher order accurate and can handle higher order excitation, it is particularly suitable for systems subject to blast loading. A large time step can be used to integrate the blast loading responses accurately. In particular, the responses at the end of the loading duration can be computed with a single time step using a sufficiently high order accurate algorithm. On the other hand, for the unconditionally stable second order accurate Newmark method, the time step size has to be kept very small in order to integrate the blast loading responses accurately. Many time steps are generally required.

The complex time step method gives accurate results at the end of a time step. Very often, it is of interest to know the responses within a time step as well. However, the evaluation of the responses within the time step by the present complex time step method has not been discussed previously.

1.2. OUTLINE

In this paper, the complex time step method is refined to give responses anywhere within a time step. The order of accuracy is shown to be only n in general, rather than $2n-1$ as at the end of the time step. A similar observation has been reported by Fung and Leung [16] for the higher order accurate time discontinuous Galerkin method and the bi-discontinuous Galerkin method. The procedure to evaluate the corresponding weighting factors α_k^* for any particular location within a time step is given. It is also found that there are some locations within a time step having an order of accuracy $n+1$ (one order higher than other locations in general).

In order to obtain accurate particular solutions, the excitation may need some modifications [14, 15]. When the excitation is expressed as a power series in time explicitly, the modification can be done easily by scaling up the coefficients. However, most of the time the explicit form is not known and a reconstruction of the polynomial may be required. The reconstruction and modification are investigated in this paper. A procedure is proposed so that the required excitations at the complex sub-step locations can be evaluated from the magnitudes of the excitation sampled at various locations within a time step interval.

In section 5, the complex time step method is used to evaluate the responses due to a high order blast loading described by a fourth order polynomial. This blast load approximation has been adopted by Singhal and Larson [5] and Singhal *et al.* [17] in evaluating flexible panel responses. In their paper, closed

form solutions were used to compute the responses and to compare with the Newmark method which was considered as the representative of time-stepping schemes. In this paper, it is shown that a sixth order accurate complex time step algorithm could predict the responses at the end of the blast loading duration accurately by using just one time step. The results are even better than those given by the Newmark method with 20 time steps. The responses within the time-step are also constructed by using the present method. A seventh order accurate algorithm which is fourth order accurate within a time step gives accurate responses within the time-step.

2. NEWMARK METHOD

The modal decomposition method can be used to uncouple (2). It is well known that the integration of the resulting system of uncoupled equations is equivalent to that for the original system. In the following, a single-degree-of-freedom system is considered. The governing equations are given by

$$u(t) + 2\xi\omega\dot{u}(t) + \omega^2 u(t) = f(t), \tag{3}$$

where ξ, ω and $f(t)$ are the damping ratio, undamped natural frequency of the system and the forcing excitation, respectively.

Given initial conditions $u(0) = u_0$, $\dot{u}(0) = v_0$ at $t = 0$ for (3), the approximate numerical solutions u_{n+1} and v_{n+1} at $t = t_{n+1}$ can be obtained from u_n and v_n at $t = t_n$ by using the following recurrence equations for the Newmark method

$$\begin{aligned} u_{n+1} &= u_n + v_n \Delta t + a_n \Delta t^2 (1 - 2\beta)/2 + a_{n+1} \Delta t^2 \beta, \\ v_{n+1} &= v_n + a_n \Delta t (1 - \gamma) + a_{n+1} \Delta t \gamma, \\ a_n &+ 2\xi\omega v_n + \omega^2 u_n = f(t_n), \\ a_{n+1} &+ 2\xi\omega v_{n+1} + \omega^2 u_{n+1} = f(t_{n+1}), \end{aligned} \tag{4}$$

where $\Delta t = t_{t+1} - t_n$ is the time step size. The parameters β and γ define the variation of acceleration over a time step and determine the stability and accuracy characteristics of the method. A satisfactory selection of these parameters is $2\beta \geqslant \gamma \geqslant 0{\cdot}5$. Two well-known selections give the constant average acceleration method ($\beta = 1/4$, $\gamma = 1/2$) and the linear acceleration method ($\beta = 1/6$, $\gamma = 1/2$).

For comparison with the analytical solutions, the Newmark algorithm is cast in the equivalent single-step two-stage form for the single-degree-of freedom system as

$$\begin{Bmatrix} u_{n+1} \\ v_{n+1} \end{Bmatrix} = [\mathbf{A}_{\mathrm{NM}}(\Delta t)] \begin{Bmatrix} u_n \\ v_n \end{Bmatrix} + [\mathbf{L}_{\mathrm{NM}}(\Delta t)] \begin{Bmatrix} f(t_n) \\ f(t_{n+1}) \end{Bmatrix}, \tag{5}$$

where $[\mathbf{A}_{\mathrm{NM}}(\Delta t)]$ is the numerical amplification matrix given by

$$\begin{bmatrix} \dfrac{2(2\beta-\gamma)\xi\omega^3\Delta t^3+(2\beta-1)\omega^2\Delta t^2+4\gamma\xi\omega\Delta t+2}{2+4\gamma\xi\omega\Delta t+2\beta\omega^2\Delta t^2} & 2\dfrac{2(2\beta-\gamma)\xi^2\omega^2\Delta t^3+(2\gamma-1)\xi\omega\Delta t^2+\Delta t}{2+4\gamma\xi\omega\Delta t+2\beta\omega^2\Delta t^2} \\ \dfrac{-(2\beta-\gamma)\omega^4\Delta t^3-2\omega^2\Delta t}{2+4\gamma\xi\omega\Delta t+2\beta\omega^2\Delta t^2} & 2\dfrac{-(2\beta-\gamma)\xi\omega^3\Delta t^3+(\beta-\gamma)\omega^2\Delta t^2+2(\gamma-1)\xi\omega\Delta t+1}{2+4\gamma\xi\omega\Delta t+2\beta\omega^2\Delta t^2} \end{bmatrix}$$

and

$$[\mathbf{L}_{\mathrm{NM}}(\Delta t)] = \begin{bmatrix} \dfrac{(1-2\beta)\Delta t^2+2\xi(\gamma-2\beta)\omega\Delta t^3}{2+4\gamma\xi\omega\Delta t+2\beta\omega^2\Delta t^2} & \dfrac{2\beta\Delta t^2}{2+4\gamma\xi\omega\Delta t+2\beta\omega^2\Delta t^2} \\ \dfrac{2(1-\gamma)\Delta t-(\gamma-2\beta)\omega^2\Delta t^3}{2+4\gamma\xi\Delta t+2\beta\omega^2\Delta t^2} & \dfrac{2\gamma\Delta t}{2+4\gamma\xi\omega\Delta t+2\beta\omega^2\Delta t^2} \end{bmatrix}.$$

Using the constant average acceleration method so that $\beta = 1/4$ and $\gamma = 1/2$, the Taylor series expansions of the entries in $[\mathbf{A}_{\mathrm{NM}}(\Delta t)]$ about Δt are

$$\begin{aligned} A_{\mathrm{NM}}(1,1) &= 1-\frac{1}{2}\omega^2\Delta t^2+\frac{1}{2}\xi\omega^3\Delta t^3-\frac{1}{8}(4\xi^2-1)\omega^4\Delta t^4+\frac{1}{4}\xi(2\xi^2-1)\omega^5\Delta t^5+\ldots, \\ A_{\mathrm{NM}}(1,2) &= \Delta t-\xi\omega\Delta t^2+\frac{1}{4}(4\xi^2-1)\omega^2\Delta t^3-\frac{1}{2}\xi(2\xi^2-1)\omega^3\Delta t^4 \\ &\quad +\frac{1}{16}(16\xi^4-12\xi^2+1)\omega^4\Delta t^5+\ldots, \\ A_{\mathrm{NM}}(2,1) &= -\omega^2\Delta t+\xi\omega^3\Delta t^2-\frac{1}{4}(4\xi^2-1)\omega^4\Delta t^3+\frac{1}{2}(2\xi^2-1)\xi\omega^5\Delta t^4 \\ &\quad -\frac{1}{16}(16\xi^4-12\xi^2+1)\omega^6\Delta t^5+\ldots, \\ A_{\mathrm{NM}}(2,2) &= 1-2\xi\omega\Delta t+\frac{1}{2}(4\xi^2-1)\omega^2\Delta t^2-(2\xi^2-1)\xi\omega^3\Delta t^3 \\ &\quad +\frac{1}{8}(16\xi^4-12\xi^2+1)\omega^4\Delta t^4-\frac{1}{8}(16\xi^4-16\xi^2+3)\xi\omega^5\Delta t^5+\ldots. \end{aligned} \tag{6}$$

The analytical amplification matrix for (3) is

$$[\mathbf{A}(t)] = \mathrm{e}^{-\xi\omega\Delta t}\begin{bmatrix} \cos(\omega_d t)+\dfrac{\xi\omega}{\omega_d}\sin(\omega_d t) & \dfrac{1}{\omega_d}\sin(\omega_d t) \\ \dfrac{\omega^2}{\omega_d}\sin(\omega_d t) & \cos(\omega_d t)-\dfrac{\xi\omega}{\omega_d}\sin(\omega_d t) \end{bmatrix}, \tag{7}$$

where $\omega_d = \omega\sqrt{1-\xi^2}$ is the damped vibration frequency. The Taylor series expansions of the entries in $[\mathbf{A}(t)]$ about t are

$$A(1,1) = 1 - \frac{1}{2}\omega^2 t^2 + \frac{1}{3}\xi\omega^3 t^3 - \frac{1}{24}(4\xi^2 - 1)\omega^4 t^4 + \frac{1}{30}\xi(2\xi^2 - 1)\omega^5 t^5 + \ldots,$$

$$A(1,2) = t - \xi\omega t^2 + \frac{1}{6}(4\xi^2 - 1)\omega^2 t^3 - \frac{1}{6}\xi(2\xi^2 - 1)\omega^3 t^4 + \frac{1}{120}(16\xi^4 - 12\xi^2 + 1)\omega^4 t^5 + \ldots,$$

$$A(2,1) = -\omega^2 t + \xi\omega^3 t^2 - \frac{1}{6}(4\xi^2 - 1)\omega^4 t^3 + \frac{1}{6}(2\xi^2 - 1)\xi\omega^5 t^4 - \frac{1}{120}(16\xi^4 - 12\xi^2 + 1)\omega^6 t^5 + \ldots,$$

$$A(2,2) = 1 - 2\xi\omega t + \frac{1}{2}(4\xi^2 - 1)\omega^2 t^2 - \frac{2}{3}(2\xi^2 - 1)\xi\omega^3 t^3 + \frac{1}{24}(16\xi^4 - 12\xi^2 + 1)\omega^4 t^4 - \frac{1}{60}(16\xi^4 - 16\xi^2 + 3)\xi\omega^5 t^5 + \ldots. \tag{8}$$

Equations (6) and (8) are useful in establishing the required conditions for higher order accurate algorithms.

3. COMPLEX TIME STEP METHOD

Fung [13–15] proposed the complex time step method by constructing a $(2n-1)$-th order accurate numerical amplification matrix $[\mathbf{A}_{2n-1}(\Delta t)]$ by

$$[\mathbf{A}_{2n-1}(\Delta t)] = \sum_{j=0}^{n} \alpha_j [\mathbf{A}_{\mathrm{NM}}(\beta_j \Delta t)], \tag{9}$$

where α_j and β_j are the weighting factors and sub-step locations, respectively. Comparing equations (6) and (8), the parameters β_0 and α_0 are chosen to be

$$\beta_0 = 0,\ \alpha_0 = \frac{1}{2}(1 + (-1)^n)\mu \tag{10}$$

and $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ are required to satisfy the following equations

$$\sum_{j=1}^{n} \alpha_j \beta_j^k = d_k \qquad \text{for } k = 1, \ldots, 2n-1, \tag{11}$$

where $d_k = 2^{k-1}/k!$ and μ is the desirable ultimate spectral radius.

It can be shown that $\beta_1, \ldots, \beta_n$ are the roots of the following nth degree polynomial

$$x^n + \Sigma_1 x^{n-1} + \Sigma_2 x^{n-2} + \ldots + \Sigma_{n-1} x + \Sigma_n = 0, \tag{12}$$

where

$$\Sigma_k = (-1)^k 2^k C_k^n \frac{(2n-1-k)!(n+(n-k)\mu)}{(2n-1)!(n+n\mu)} \qquad \text{and} \qquad C_k^n = \frac{n!}{(n-k)!k!}. \quad (13)$$

$\alpha_1, \alpha_2, \ldots, \alpha_n$ can be determined from the first n equations in (11) after $\beta_1, \ldots, \beta_n$ are determined. The resultant algorithms are unconditionally stable if $-1 < \mu \leqslant 1$. The order of accuracy is improved from $2n-1$ to $2n$ if $\mu = 1$. However, this higher order accuracy is achieved at the end of the time step only.

In actual computation, the numerical amplification matrices are not computed. Instead, the complex responses U_j and V_j are computed and combined as follows:

$$u_1 = \sum_{j=0}^{n} \alpha_j U_j \qquad \text{and} \qquad v_1 = \sum_{j=0}^{n} \alpha_j V_j, \quad (14)$$

where U_j and V_j are responses at the sub-step locations $\beta_j \Delta t$ computed using the Newmark method in equations (4) and (5).

3.1. WEIGHTING PARAMETERS FOR RESPONSES WITHIN A TIME STEP

u_1 and v_1 in equation (14) are responses at the end of a time step only. In the following, the responses at any time $\eta \Delta t$ ($0 \leqslant \eta \leqslant 1$) within the time interval are considered. Equivalently, a numerical amplification matrix $[\mathbf{A}_s(\eta \Delta t)]$ at $\eta \Delta t$ is to be constructed from $[\mathbf{A}_{\mathrm{NM}}(\beta_j \Delta t)]$. In other words, it is required to construct

$$[\mathbf{A}_s(\eta \Delta t)] = \sum_{j=0}^{n} \alpha_j^* [\mathbf{A}_{\mathrm{NM}}(\beta_j \Delta t)], \quad (15)$$

where $\beta_1, \ldots, \beta_n$ have been determined previously in equation (12) and $\alpha_1^*, \ldots, \alpha_n^*$ would depend on η in general.

By comparing the Taylor series expansions of equation (15) and $[\mathbf{A}(\eta \Delta t)]$, it can be shown that the required conditions for an sth order accurate $[\mathbf{A}_s(\eta \Delta t)]$ are

$$\sum_{j=0}^{n} \alpha_j^* = 1, \qquad \sum_{j=1}^{n} \alpha_j^* \beta_j^k = d_k \eta^k \quad \text{for} \quad k = 1, \ldots, s \quad (16, 17)$$

Since there are $(n+1)$ undetermined parameters $(\alpha_0^*, \alpha_1^*, \ldots, \alpha_n^*)$, s can at least be n. In other words, the responses within a time step can be nth order accurate in general. In matrix notation, the weighting parameters α_j^* at any particular time $\eta \Delta t$ within a time interval can be obtained from

$$\begin{bmatrix} \beta_1 & \beta_2 & \ldots & \beta_n \\ \beta_1^2 & \beta_2^2 & \ldots & \beta_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^n & \beta_2^n & \ldots & \beta_n^n \end{bmatrix} \begin{Bmatrix} \alpha_1^* \\ \alpha_2^* \\ \vdots \\ \alpha_n^* \end{Bmatrix} = \begin{Bmatrix} d_1 \eta \\ d_2 \eta^2 \\ \vdots \\ d_n \eta^n \end{Bmatrix} \quad (18)$$

and $\alpha_0^* = 1 - \alpha_1^* - \alpha_2^* - \ldots - \alpha_n^*$ since $\beta_0 = 0$. Obviously, when $\eta = 1$, α_j^* would be

equal to those α_j in equation (11) and the order of accuracy would be at least $2n-1$. It can be proved that α_j^* can be given explicitly as

$$\alpha_j^* = \frac{1}{2}\frac{\displaystyle\sum_{i=1}^{n}\left(\sum_{k=0}^{n-i} 2^i \Sigma_k \beta_j^{n-i-zk-1}\right)\frac{\eta^i}{i!}}{(\beta_j-\beta_1)\cdots(\beta_j-\beta_{j-1})(\beta_j-\beta_{j+1})\cdots(\beta_j-\beta_n)}. \tag{19}$$

If the responses at n regular intervals are of interest (i.e., $\eta = k/n$ for $k = 1, \ldots, n$), α_{jk}^* corresponding to $(k/n)\Delta t$ can be put into a matrix form as

$$[\alpha^*] = [\beta]^{-1}[D][Q] \tag{20}$$

where

$$[\alpha^*] = \begin{bmatrix} \alpha_{11}^* & \alpha_{12}^* & \ldots & \alpha_{1n}^* \\ \alpha_{21}^* & \alpha_{22}^* & \ldots & \alpha_{2n}^* \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1}^* & \alpha_{n2}^* & \ldots & \alpha_{nn}^* \end{bmatrix}, \quad [\beta] = \begin{bmatrix} \beta_1 & \beta_2 & \ldots & \beta_n \\ \beta_1^2 & \beta_2^2 & \ldots & \beta_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^n & \beta_2^n & \ldots & \beta_n^n \end{bmatrix},$$

$$[\mathbf{D}] = \begin{bmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_n \end{bmatrix} \quad \text{and} \quad [\mathbf{Q}] = \begin{bmatrix} \left(\frac{1}{n}\right) & \left(\frac{2}{n}\right) & \ldots & \left(\frac{n-1}{n}\right) & 1 \\ \left(\frac{1}{n}\right)^2 & \left(\frac{2}{n}\right)^2 & \ldots & \left(\frac{n-1}{n}\right)^2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \left(\frac{1}{n}\right)^n & \left(\frac{2}{n}\right)^n & \ldots & \left(\frac{n-1}{n}\right)^n & 1 \end{bmatrix}.$$

Some typical values of of α_j^* for $n = 2$, 3 and 4 are shown in Tables 1–3.

Examining equations (18) and (19) reveals that α_j^* is at most an nth degree polynomial of η. Hence, the responses at other locations can in fact be obtained

TABLE 1

Weighting parameters for interpolation: third order ($n = 2$) $\beta_0 = 0$, $\alpha_0^* = 1 - \alpha_1^* - \alpha_2^*$

				$\eta = 1/2$	$\eta = 1$
$\mu = 0$	$\mathrm{Re}(\beta_1^*)$, $\mathrm{Re}(\beta_2^*)$	0·666666667	$\mathrm{Re}(\alpha_1^*)$, $\mathrm{Re}(\alpha_2^*)$	0·312500000	0·250000000
	$\mathrm{Im}(\beta_1^*)$, $-\mathrm{Im}(\beta_2^*)$	0·471405208	$\mathrm{Im}(\alpha_1^*)$, $-\mathrm{Im}(\alpha_2^*)$	−0·088388348	−0·707106781
$\mu = 1/2$	$\mathrm{Re}(\beta_1^*)$, $\mathrm{Re}(\beta_2^*)$	0·555555556	$\mathrm{Re}(\alpha_1^*)$, $\mathrm{Re}(\alpha_2^*)$	0·343750000	0·125000000
	$\mathrm{Im}(\beta_1^*)$, $-\mathrm{Im}(\beta_2^*)$	0·368513866	$\mathrm{Im}(\alpha_1^*)$, $-\mathrm{Im}(\alpha_2^*)$	−0·160177902	−1·168356460
$\mu = 1$	$\mathrm{Re}(\beta_1^*)$, $\mathrm{Re}(\beta_2^*)$	0·500000000	$\mathrm{Re}(\alpha_1^*)$, $\mathrm{Re}(\alpha_2^*)$	0·375000000	0·000000000
	$\mathrm{Im}(\beta_1^*)$, $-\mathrm{Im}(\beta_2^*)$	0·288675135	$\mathrm{Im}(\alpha_1^*)$, $-\mathrm{Im}(\alpha_2^*)$	−0·216506351	−1·732050808

TABLE 2

Weighting parameters for interpolation: fifth order ($n=3$) $\beta_0=0$, $\alpha_0^*=1-\alpha_1^*-\alpha_2^*-\alpha_3^*$

				$\eta=1/3$	$\eta=2/3$	$\eta=1$
$\mu=0$	β_1^*	0·549777659	α_1^*	0·323240754	0·681969268	2·514888929
	$\mathrm{Re}(\beta_2^*)$, $\mathrm{Re}(\beta_3^*)$	0·325111170	$\mathrm{Re}(\alpha_2^*)$, $\mathrm{Re}(\alpha_3^*)$	0·180972216	−0·100243893	−1·007444464
	$\mathrm{Im}(\beta_2^*)$, $-\mathrm{Im}(\beta_3^*)$	0·369898649	$\mathrm{Im}(\alpha_2^*)$, $-\mathrm{Im}(\alpha_3^*)$	−0·051298661	−0·482451097	−0·368253778
$\mu=1/2$	β_1^*	0·480424464	α_1^*	0·329280683	0·935797197	4·117654848
	$\mathrm{Re}(\beta_2^*)$, $\mathrm{Re}(\beta_3^*)$	0·293121101	$\mathrm{Re}(\alpha_2^*)$, $\mathrm{Re}(\alpha_3^*)$	0·182581881	−0·273454154	−1·683827424
	$\mathrm{Im}(\beta_2^*)$, $-\mathrm{Im}(\beta_3^*)$	0·314804051	$\mathrm{Im}(\alpha_2^*)$, $-\mathrm{Im}(\alpha_3^*)$	−0·108165117	−0·599415993	−0·014149399
$\mu=1$	β_1^*	0·430628846	α_1^*	0·318334440	1·164812864	6·158265977
	$\mathrm{Re}(\beta_2^*)$, $\mathrm{Re}(\beta_3^*)$	0·284685577	$\mathrm{Re}(\alpha_2^*)$, $\mathrm{Re}(\alpha_3^*)$	0·192684632	−0·434258284	−2·579132989
	$\mathrm{Im}(\beta_2^*)$, $-\mathrm{Im}(\beta_3^*)$	0·271599851	$\mathrm{Im}(\alpha_2^*)$, $-\mathrm{Im}(\alpha_3^*)$	−0·159315753	−0·759055620	−0·337708298

TABLE 3

Weighting parameters for interpolation: seventh order ($n=4$) $\beta_0=0$, $\alpha_0^*=1-\alpha_1^*-\alpha_2^*-\alpha_3^*-\alpha_4^*$

				$\eta=1/4$	$\eta=1/2$	$\eta=3/4$	$\eta=1$
$\mu=0$	$\mathrm{Re}(\beta_1^*)$, $\mathrm{Re}(\beta_2^*)$	0·377327607	$\mathrm{Re}(\alpha_1^*)$, $\mathrm{Re}(\alpha_2^*)$	0·221415691	0·326316246	0·899463708	0·600688568
	$\mathrm{Im}(\beta_1^*)$, $-\mathrm{Im}(\beta_2^*)$	0·123548834	$\mathrm{Im}(\alpha_1^*)$, $-\mathrm{Im}(\alpha_2^*)$	−0·031819636	−0·428147098	−1·598641414	−6·077528225
	$\mathrm{Re}(\beta_3^*)$, $\mathrm{Re}(\beta_4^*)$	0·194100965	$\mathrm{Re}(\alpha_3^*)$, $\mathrm{Re}(\alpha_4^*)$	0·119404622	−0·123191246	−0·668018396	−0·350688568
	$\mathrm{Im}(\beta_3^*)$, $-\mathrm{Im}(\beta_4^*)$	0·288364942	$\mathrm{Im}(\alpha_3^*)$, $-\mathrm{Im}(\alpha_4^*)$	−0·049749168	−0·339452973	0·111801925	1·419933060
$\mu=1/2$	$\mathrm{Re}(\beta_1^*)$, $\mathrm{Re}(\beta_2^*)$	0·341851489	$\mathrm{Re}(\alpha_1^*)$, $\mathrm{Re}(\alpha_2^*)$	0·201472435	0·412942036	1·118352301	−0·253820656
	$\mathrm{Im}(\beta_1^*)$, $-\mathrm{Im}(\beta_2^*)$	0·109060021	$\mathrm{Im}(\alpha_1^*)$, $-\mathrm{Im}(\alpha_2^*)$	−0·071811480	−0·546018650	−2·394986060	−10·049013892
	$\mathrm{Re}(\beta_3^*)$, $\mathrm{Re}(\beta_4^*)$	0·181958034	$\mathrm{Re}(\alpha_3^*)$, $\mathrm{Re}(\alpha_4^*)$	0·130070533	−0·233254536	−0·841496833	0·378820656
	$\mathrm{Im}(\beta_3^*)$, $-\mathrm{Im}(\beta_4^*)$	0·255958435	$\mathrm{Im}(\alpha_3^*)$, $-\mathrm{Im}(\alpha_4^*)$	−0·096215652	−0·358374129	0·450817416	2·258593898
$\mu=1$	$\mathrm{Re}(\beta_1^*)$, $\mathrm{Re}(\beta_2^*)$	0·316867519	$\mathrm{Re}(\alpha_1^*)$, $\mathrm{Re}(\alpha_2^*)$	0·164719866	1·485193026	0·435050896	1·143874647
	$\mathrm{Im}(\beta_1^*)$, $-\mathrm{Im}(\beta_2^*)$	0·094882025	$\mathrm{Im}(\alpha_1^*)$, $-\mathrm{Im}(\alpha_2^*)$	−0·127757184	−0·660494516	−3·247492038	−15·298158259
	$\mathrm{Re}(\beta_3^*)$, $\mathrm{Re}(\beta_4^*)$	0·183132481	$\mathrm{Re}(\alpha_3^*)$, $\mathrm{Re}(\alpha_4^*)$	0·157545759	−0·328943026	−1·112785271	1·143874647
	$\mathrm{Im}(\beta_3^*)$, $-\mathrm{Im}(\beta_4^*)$	0·231325226	$\mathrm{Im}(\alpha_3^*)$, $-\mathrm{Im}(\alpha_4^*)$	−0·137607209	−0·405615877	0·795687159	3·452040791

by interpolating the initial conditions and the n determined responses. The order of accuracy of the interpolated responses is of order at least n in general.

3.2. LOCATIONS WITH ONE ORDER HIGHER IN ACCURACY

From equation (17), it can be seen that it may be possible to choose η so that an additional equation with $k = n + 1$ is satisfied, i.e.,

$$[\beta_1^{n+1} \quad \beta_2^{n+1} \quad \dots \quad \beta_n^{n+1}] \begin{Bmatrix} \alpha_1^* \\ \alpha_2^* \\ \\ \vdots \\ \alpha_n^* \end{Bmatrix} = d_{n+1}\eta^{n+1}. \tag{21}$$

Using equation (18), equation (21) can be written as

$$[\beta_1^n \quad \beta_2^n \quad \dots \quad \beta_n^n] \begin{bmatrix} 1 & 1 & \dots & 1 \\ \beta_1 & \beta_2 & \dots & \beta_n \\ & & & \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^{n-1} & \beta_2^{n-1} & \dots & \beta_n^{n-1} \end{bmatrix}^{-1} \begin{Bmatrix} d_1\eta \\ d_2\eta^2 \\ \\ \vdots \\ d_n\eta^n \end{Bmatrix} = d_{n+1}\eta^{n+1}. \tag{22}$$

Since $\beta_1, \dots, \beta_n$ are roots of equation (12), equation (22) can be shown to be equal to

$$[\Sigma_n \quad \Sigma_{n-1} \quad \dots \quad \Sigma_1] \begin{Bmatrix} d_1\eta \\ d_2\eta^2 \\ \\ \vdots \\ d_n\eta^n \end{Bmatrix} = -d_{n+1}\eta^{n+1}. \tag{23}$$

Therefore, the locations with one order higher in accuracy are given by the roots of the following polynomial

$$d_{n+1}\eta^n + d_n\Sigma_1\eta^{n-1} + \dots + d_2\Sigma_{n-1}\eta + d_1\Sigma_n = 0. \tag{24}$$

The locations with one order higher in accuracy can therefore be found systematically for given n. The accuracy of the responses at these locations is at least $n+1$. The locations and the corresponding weighting factors for $n = 2$, 3 and 4 are as shown in Tables 4–6.

4. EVALUATION OF EXCITATION AT COMPLEX SUB-STEP LOCATIONS

The excitation force $f(t)$ in equation (3) in general can be approximated by a polynomial function or can be expanded into Taylor series at the beginning of a time step Δt as

TABLE 4

Locations and weighting factors with order of accuracy $n+1$ within a time step, where $n=2$

μ	$n=2$ η	$\mathrm{Re}(\alpha_1^*)$ $\mathrm{Re}(\alpha_2^*)$	$\mathrm{Im}(\alpha_1^*)$ $-\mathrm{Im}(\alpha_2^*)$
0	–	–	–
0·5	0·666666667	0·333333333	−0·402015126
1·0	0·500000000	0·375000000	−0·216506351

$$f(t) = f(0) + f'(0)t + \frac{1}{2}f''(0)t^2 + \frac{1}{3!}f'''(0)t^3 + \ldots + \frac{1}{n!}f^{(n)}(0)t^n + \ldots \quad \text{for} \quad 0 \leqslant t \leqslant \Delta t. \tag{25}$$

If the truncated Taylor series is used to represent the excitation, the terms retained should conform to the required accuracy. It has been shown that to maintain the accuracy of the numerical solutions for a given mth order accurate algorithm, only excitation terms from t^0 to t^{m-1} are required.

In the complex time step method, it has been shown [14] that the given excitation in the form

$$f(t) = f_0 + f_1 t + f_2 t^2 + f_3 t^3 + f_4 t^4 + f_5 t^5 + f_6 t^6 + + f_7 t^7 + \ldots = f_0 + \sum_{k=1} f_k t^k \tag{26}$$

has to be modified to the form

$$\bar{f}(t) = f_0 + f_1 t + f_2 t^2 + \frac{3}{2} f_3 t^3 + 3 f_4 t^4 + \frac{15}{2} f_5 t^5 + \frac{45}{2} f_6 t^6 + \frac{315}{4} f_7 t^7 + \ldots = f_0 + \sum_{k=1} \frac{f_k}{d_k} t^k \tag{27}$$

where d_k are defined in equation (11).

TABLE 5

Locations and weighting factors with order of accuracy $n+1$ within a time step, where $n=3$

μ	$n=3$ η	α_1^*	$\mathrm{Re}(\alpha_2^*)$ $\mathrm{Re}(\alpha_3^*)$	$\mathrm{Im}(\alpha_2^*)$ $-\mathrm{Im}(\alpha_3^*)$
0	0·400000000	0·346108927	0·166945536	−0·136747359
0·5	0·333333333	0·329280683	0·182581881	−0·108165117
0·5	0·800000000	1·770805424	−0·705402712	−0·576229233
1·0	0·276393202	0·326575082	0·198515858	−0·041847130
1·0	0·723606798	1·624084090	−0·673845444	−0·750916276

TABLE 6

Locations and weighting factors with order of accuracy $n+1$ within a time step, where $n=4$

μ	$n=4$ η	$\mathrm{Re}(\alpha_1^*)$ $\mathrm{Re}(\alpha_2^*)$	$\mathrm{Im}(\alpha_1^*)$ $-\mathrm{Im}(\alpha_2^*)$	$\mathrm{Re}(\alpha_3^*)$ $\mathrm{Re}(\alpha_4^*)$	$\mathrm{Im}(\alpha_3^*)$ $-\mathrm{Im}(\alpha_4^*)$
0	0·226540920	0·230277199	−0·002447173	0·115553873	−0·012654284
0	0·630601937	0·607145801	−0·832296999	−0·417991451	−0·223714146
0·5	0·198385342	0·118655703	0·001808796	0·230523905	0·006782061
0·5	0·553793169	−0·386516761	−0·292223635	0·564341262	−0·729033505
0·5	0·866869108	−0·693420760	1·225249906,	1·006822654	−4·850881672
1·0	0·172673165	0·243876683	0·033161340	0·113266174	0·036901802
1·0	0·500000000	0·485193026	−0·660494516	−0·328943026	−0·405615877
1·0	0·827326835	1·332821980	−5·463507482	−0·975679122	1·505995257

It is not convenient to express the forcing function as a power series as in equation (26) and then modify the coefficients accordingly as in equation (27) before evaluating the modified forcing excitation at the complex sub-step location $\beta_j \Delta t$. In the following, a procedure is proposed to compute the forcing excitation at the complex sub-step locations directly from the force magnitudes sampled at discrete locations within the time step.

Figure 4 shows the magnitude of the forcing excitations f_j at a particular time $t_j = \eta_j \Delta t$. If the forcing function is to be approximated by a polynomial of degree p, the required polynomial can be written as

$$f(\eta) = f_0 + a_1\eta + a_2\eta^2 + \cdots + a_p\eta^p, \tag{28}$$

where $a_1, \ldots, a_p$ are unknown coefficients to be determined. If $f_j = f(\eta_j)$ corresponds to the excitation at $\eta_j \Delta t$, then

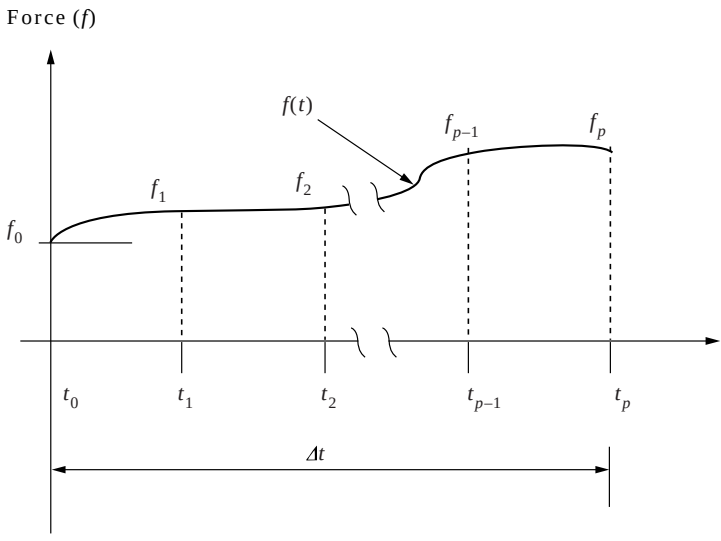

Figure 4. Forcing excitation.

$$\begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_p \end{Bmatrix} = \begin{Bmatrix} f_0 \\ f_0 \\ \vdots \\ f_0 \end{Bmatrix} + \begin{bmatrix} \eta_1 & \eta_1^2 & \dots & \eta_1^p \\ \eta_2 & \eta_2^2 & \dots & \eta_2^p \\ \vdots & \vdots & \ddots & \vdots \\ \eta_p & \eta_p^2 & \dots & \eta_p^p \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{Bmatrix} \quad \text{or} \quad \{\mathbf{a}\} = [\bar{\eta}]^{-1}(\{\mathbf{f}\} - \{\mathbf{f}_0\}). \tag{29}$$

From equation (27), the modified excitation will then be in the form

$$\bar{f}(\eta) = f_0 + \frac{a_1}{d_1}\eta + \frac{a_2}{d_2}\eta^2 + \ldots + \frac{a_p}{d_p}\eta^p. \tag{30}$$

As a result, the modified forcing excitation can be written as

$$\bar{f}(\eta) = f_0 + [\eta \quad \eta^2 \quad \dots \quad \eta^p][\mathbf{D}]^{-1}[\bar{\eta}]^{-1}(\{\mathbf{f}\} - \{\mathbf{f}_0\}). \tag{31}$$

The excitations at the complex time step locations $\beta_1 \Delta t, \ldots, \beta_n \Delta t$ can be given collectively as

$$\begin{Bmatrix} \bar{f}(\beta_1) \\ \bar{f}(\beta_2) \\ \vdots \\ \bar{f}(\beta_n) \end{Bmatrix} = \{\mathbf{f}_0\} + \begin{bmatrix} \beta_1 & \beta_1^2 & \dots & \beta_1^p \\ \beta_2 & \beta_2^2 & \dots & \beta_2^p \\ \vdots & \vdots & \ddots & \vdots \\ \beta_n & \beta_n^2 & \dots & \beta_n^p \end{bmatrix} \begin{bmatrix} \frac{1}{d_1} & 0 & \dots & 0 \\ 0 & \frac{1}{d_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{d_p} \end{bmatrix} [\bar{\eta}]^{-1}(\{\mathbf{f}\} - \{\mathbf{f}_0\}). \tag{32}$$

It is interesting to note that if $p = n$, and regular sampling intervals are used (i.e., $\eta_k = k/n$) then

$$[\bar{\eta}] = [\mathbf{Q}]^{\mathrm{T}} \tag{33}$$

and equation (32) becomes

$$\{\bar{\mathbf{f}}\} = \{\mathbf{f}_0\} + [\beta]^{\mathrm{T}}[\mathbf{D}]^{-1}[\mathbf{Q}]^{-\mathrm{T}}\{\mathbf{f}\} = \{\mathbf{f}_0\} + [\alpha^*]^{-\mathrm{T}}(\{\mathbf{f}\} - \{\mathbf{f}_0\}). \tag{34}$$

5. NUMERICAL EXAMPLE: FOURTH ORDER BLAST LOADING FUNCTION

Since the blast loading is generally described by higher order excitation functions, higher order accurate algorithms can be used to compute the responses more accurately even if a large time step is used. In the following, the complex time step method is used to evaluate the responses due to a high order blast loading function described in Figure 1.

5.1. BLAST LOADING FUNCTION

Fansler and Schmidt [18] and Heap *et al.* [19] showed that the free-field overpressures from gun blast based on experiments and field tests can be described by a fourth order blast loading function with a duration of T_s as

$$p(t) = \bar{p}\left(1 - \frac{t - t_a}{\tau}\right)\left(1 - \frac{\alpha(t - t_a)}{\tau}\right)\left(1 - \left(\frac{\alpha(t - t_a)}{\tau}\right)^2\right), \tag{35}$$

where $\bar{p}$, t, t_a and τ are defined in Figure 1. The positive phase duration τ generally varies between 5×10^{-4} and 15×10^{-4} s and the remaining parameters are $\alpha = 0{\cdot}2728$ and $T_s = \tau/\alpha = 3{\cdot}667\tau$. Without loss of generality, $\bar{p}$ can be assumed to be unity and $t_a = 0$.

Since the blast loading function is a fourth degree polynomial in t, Taylor series expansion or a polynomial function up to the fourth degree can be used to represent the blast loading function in equation (35) exactly. In this case, $p = 4$ in equation (28) would be sufficient. Within a time interval, five forcing magnitudes are evaluated at various locations and the excitations at the complex sub-step locations are calculated from equation (32).

5.2. WEIGHTING PARAMETERS

Consider the equation of motion of a single-degree-of-freedom system in the form

$$\ddot{u}(t) + 2\xi\omega\dot{u}(t) + \omega^2 u(t) = p(t), \tag{36}$$

where $\xi = 0$ and $\omega = 0{\cdot}9$ as in Singhal and Larson [5]. The blast loading $p(t)$ is defined in equation (35) with $\tau = 15 \times 10^{-4}$ s. Since the blast loading is a fourth order function, a fifth or higher order accurate algorithm can be used to predict the responses accurately.

In the present formulation, the first step is to evaluate the algorithmic parameters α_j and β_j. Consider a fifth order accurate complex time step algorithm with $n = 3$ and $\mu = 0$. The sub-stepping procedure is shown in Figure 5(a). The required algorithmic parameters are

$$\beta_0 = 0, \qquad \alpha = \frac{1}{2} \tag{37}$$

and β_1, β_2 and β_3 are the roots of

$$\beta^3 - \frac{6}{5}\beta^2 + \frac{3}{5}\beta - \frac{2}{15} = 0 \tag{38}$$

and are given by

$$\begin{aligned}
\beta_1 &= \tfrac{1}{5}(3^{1/3}) - \tfrac{1}{15}(3^{2/3}) + \tfrac{2}{5}, \\
\beta_2 &= -(\tfrac{1}{10}(3^{1/3}) + \tfrac{1}{30}(3^{2/3}) + \tfrac{2}{5}) + \mathrm{i}\,\tfrac{1}{2}\sqrt{3}(\tfrac{1}{5}(3^{1/3}) + \tfrac{1}{15}(3^{2/3})), \\
\beta_3 &= -(\tfrac{1}{10}(3^{1/3}) + \tfrac{1}{30}(3^{2/3}) + \tfrac{2}{5}) - \mathrm{i}\,\tfrac{1}{2}\sqrt{3}(\tfrac{1}{5}(3^{1/3}) + \tfrac{1}{15}(3^{2/3})).
\end{aligned} \tag{39}$$

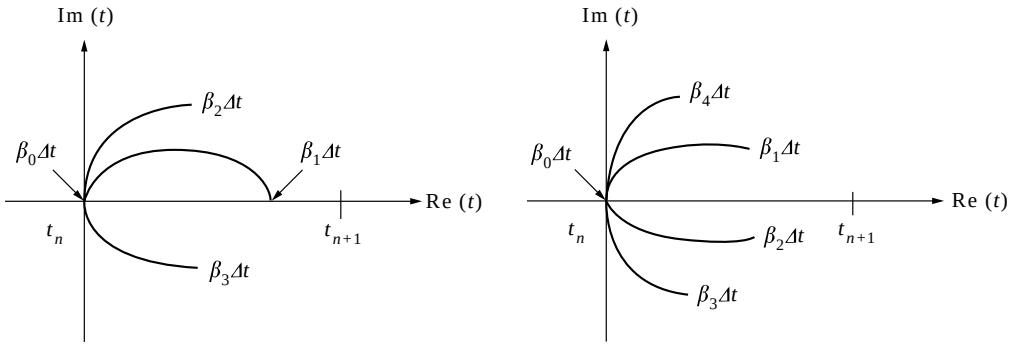

Figure 5. Sub-stepping schemes for higher order algorithms: (a) fifth and sixth order algorithms, (b) seventh order algorithm.

Similarly, α_1, α_2 and α_3 can be evaluated from

$$\begin{bmatrix} 1 & 1 & 1 \\ \beta_1 & \beta_2 & \beta_3 \\ \beta_1^2 & \beta_2^2 & \beta_3^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{Bmatrix} \tag{40}$$

and are given by

$$\begin{aligned} \alpha_1 &= \tfrac{1}{6} + \tfrac{2}{3}(3^{1/3}) + \tfrac{2}{3}(3^{2/3}), \\ \alpha_2 &= \tfrac{1}{6} - \tfrac{1}{3}(3^{1/3}) - \tfrac{1}{3}(3^{2/3}) + \mathrm{i}(\tfrac{1}{3}(3^{5/6}) - 3^{1/6}), \\ \alpha_3 &= \tfrac{1}{6} - \tfrac{1}{3}(3^{1/3}) - \tfrac{1}{3}(3^{2/3}) - \mathrm{i}(\tfrac{1}{3}(3^{5/6}) - 3^{1/6}). \end{aligned} \tag{41}$$

5.3. RESPONSES AT THE END OF THE FIRST TIME STEP

The evaluated algorithmic parameters α_j and β_j for the fifth order accurate algorithm are now used to solve for the responses of the blast loading at the end of a time step. Consider the first time step of size Δt ($0 \leqslant \Delta t \leqslant T_s$). The responses for the displacement U_j and the velocity V_j at each sub-step locations $\beta_j\Delta t$ are evaluated independently using the Newmark method in equation (4) at $\beta_j\Delta t$. Note that since β_2 and β_3 are complex conjugates, the corresponding results are also complex conjugates. Hence, only one of them has to be evaluated. If the computational effort of complex number multiplication is assumed to be four times that of real number multiplication, the computational effort of the present fifth order algorithms would be five times that of the Newmark method (since one real time step $\beta_1\Delta t$ and one complex time step $\beta_2\Delta t$ or $\beta_3\Delta t$ are evaluated).

By linearly combining the sub-step responses using the weighting factors α_j in equation (41), the responses at the end of a chosen time step Δt are given by

$$\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \sum_{k=0}^{3} \alpha_k \begin{Bmatrix} U_k \\ V_k \end{Bmatrix}, \quad \text{where} \quad U_2 = \bar{U}_3 \quad \text{and} \quad V_2 = \bar{V}_3. \tag{42}$$

Note that U_0 and V_0 correspond to the sub-step time at $\beta_0 \Delta t = 0$. These are the initial conditions and therefore need not be evaluated, i.e., $U_0 = u_0 = 0$ and $V_0 = v_0 = 0$.

For comparisons with the analytical solutions, the Taylor series expansion of u_1 about Δt is found to be

$$u_1 = \frac{1}{2}\Delta t^2 - \frac{1}{6}\frac{\alpha+1}{\tau}\Delta t^3 - \frac{1}{24}\frac{\omega^2\tau^2 + 2\alpha^2 - 2\alpha}{\tau^2}\Delta t^4 + \frac{1}{120}\frac{6\alpha^3 + 6\alpha^2 + \alpha\omega^2\tau^2 + \omega^2\tau^2}{\tau^3}\Delta t^5 + \frac{11}{7200}\frac{-24\alpha^3 + 2\alpha^2\omega^2\tau^2 + \omega^4\tau^4 - 2\alpha\omega^2\tau^2}{\tau^4}\Delta t^6 + \ldots. \tag{43}$$

Similarly, the Taylor series expansion of the analytical solution of equation (36) is found to be

$$u_{\text{exact}} = \frac{1}{2}\Delta t^2 - \frac{1}{6}\frac{\alpha+1}{\tau}\Delta t^3 - \frac{1}{24}\frac{\omega^2\tau^2 + 2\alpha^2 - 2\alpha}{\tau^2}\Delta t^4 + \frac{1}{120}\frac{6\alpha^3 + 6\alpha^2 + \alpha\omega^2\tau^2 + \omega^2\tau^2}{\tau^3}\Delta t^5 + \frac{1}{720}\frac{-24\alpha^3 + 2\alpha^2\omega^2\tau^2 + \omega^4\tau^4 - 2\alpha\omega^2\tau^2}{\tau^4}\Delta t^6 + \ldots. \tag{44}$$

From the above, it can be verified that the algorithm is fifth order accurate for the α_j and β_j values used and the truncation error is

$$\text{Error}[O(\Delta t^6)] = \frac{1}{7200}\frac{-24\alpha^3 + 2\alpha^2\omega^2\tau^2 + \omega^4\tau^4 - 2\alpha\omega^2\tau^2}{\tau^4}\Delta t^6. \tag{45}$$

The absolute magnitude of the error of course depends on the actual time step size Δt. Table 7 shows the relative differences between the numerical results and the analytical results for the displacement at the end of the first time step for various Δt. It can be seen that very good accuracy can be obtained when the time step size is about half of the duration of the blast loading. The accuracy starts to deteriorate when the time step size increases further. Therefore, to find the responses at $t = T_s$, two or more time steps should be used for the present fifth order accurate algorithm. Alternatively, sixth or higher order accurate algorithms could be used. The accuracy of the velocity response is not shown in Table 7 since a very close agreement with the exact solutions is obtained throughout the range of time steps under consideration. It can be seen from Table 8 that for the velocity response, even if $\Delta t = T_s$, up to 4 significant figures coincide with the exact solution when the fifth order complex time step method is used.

TABLE 7

Displacement for the first time step with various Δt using the fifth order accurate algorithm ($n=3$, $\mu=0$)

$\Delta t/T_s$	Complex time step (10^{-6} m)	Exact (10^{-6} m)	Relative difference
0	0	0	–
0·05	0·034921613	0·034921619	−1·72E-07
0·10	0·128481122	0·128481492	−2·88E-06
0·15	0·264837012	0·264841229	−1·59E-05
0·20	0·429587299	0·429610993	−5·52E-05
0·25	0·609919953	0·610010339	−1·48E-04
0·30	0·794717517	0·794987408	−3·39E-04
0·35	0·974615903	0·975296467	−6·98E-04
0·40	1·142017373	1·143533797	−1·33E-03
0·45	1·291057715	1·294131941	−2·38E-03
0·50	1·417527591	1·423312287	−4·06E-03
0·55	1·518748082	1·528996019	−6·70E-03
0·60	1·593400409	1·610673400	−1·07E-02
0·65	1·641309847	1·669231421	−1·67E-02
0·70	1·663183819	1·706739787	−2·55E-02
0·75	1·660304182	1·726195264	−3·82E-02
0·80	1·634173691	1·731224370	−5·61E-02
0·85	1·586116659	1·725744417	−8·09E-02
0·90	1·516833790	1·713582904	−1·15E-01
0·95	1·425911215	1·698055262	−1·60E-01
1·00	1·311283698	1·681500944	−2·20E-01

5.4. RESPONSES AT $t=T_s$

The calculation of the responses at the end of the loading duration T_s is also carried out by using 2, 3 and 4 time steps ($\Delta t=T_s/2$, $\Delta t=T_s/3$ and $\Delta t=T_s/4$, respectively) using the present fifth order accurate algorithm. As shown in Table 8, the results approach to the analytical solutions.

The sixth and seventh order accurate algorithms are also used to evaluate the responses with only one time step ($\Delta t=T_s$). The sixth order accurate algorithm is computed using $n=3$ and $\mu=1$ while the seventh order accurate algorithm uses $n=4$ and $\mu=0$. The sub-stepping procedure is illustrated in Figure 5. It can be seen from Table 8 that the displacements calculated from the sixth and seventh order accurate algorithms have 7 and 8 significant figures respectively.

On the other hand, the Newmark method using 20 time steps produces solutions with 2 significant figures only. In order to achieve the same accuracy as the sixth order accurate complex time step algorithm, about 1400 time steps are required. For the velocity response, the higher order algorithms give good predictions as well but not for the Newmark method with 20 time steps.

Table 8 also shows the computational effort relative to the Newmark method. It can be seen that the computational efforts for the present algorithms are much lower than the Newmark method for accurate solutions.

TABLE 8

Reuls at $t = T_s$ by various methods and time steps

Method	n	μ	No. of time step	Effort relative to Newmark method	Displacement (10^{-6} m)	Relative difference	Velocity (10^{-5} m/s)	Relative difference
Exact	–	–	–	–	1·681500944	–	−6·047288192	–
Newmark method	–	–	20	20	1·698308634	1·00E-02	−5·512455909	−8·84E-02
Newmark method	–	–	1400	1400	1·681504378	2·04E-06	−6·047179129	−1·80E-05
Present 5th order	3	0	1	5	1·311283698	−2·20E-01	−6·047147435	−2·33E-05
Present 5th order	3	0	2	10	1·669931627	−6·88E-03	−6·047284254	−6·51E-07
Present 5th order	3	0	3	15	1·679977413	−9·06E-04	−6·047287590	−9·95E-08
Present 5th order	3	0	4	20	1·681 139403	−2·15E-04	−6·047288076	−1·92E-08
Present 6th order	3	1	1	5	1·681501386	2·63E-07	−6·047276410	−1·95E-06
Present 7th order	4	0	1	8	1·681500898	−2·74E-08	−6·047288192	0·00E+00

5.5. RESPONSES WITHIN A TIME STEP

The complex time step method can be used to calculate the responses within a time step after the responses at the sub-step locations are computed. For the fifth order accurate algorithm, the values of β_j from equation (39) are substituted into

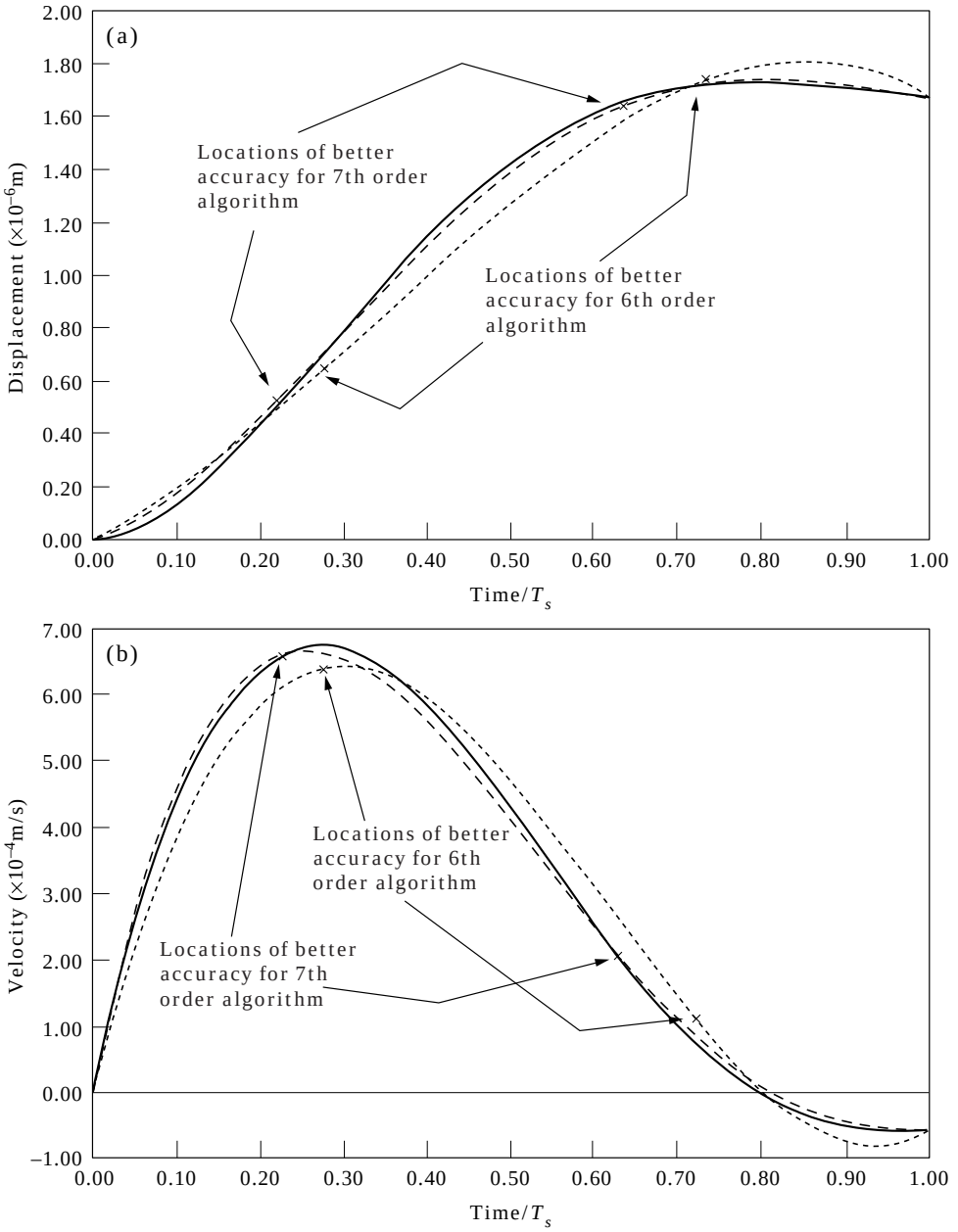

Figure 6. (a) Interpolation of displacement. (b) Interpolation of velocity. ———, Exact 6th order (2 time steps); - - - - -, 6th order; – – –, 7th order.

equation (18) or (19) to yield the required weighting factors for the response at various $\eta\Delta t$ where $0 \leqslant \eta \leqslant 1$.

Figures 6(a) and (b) show the interpolated displacement and velocity respectively for the sixth and seventh order accurate algorithms. The sixth order accurate solution does not follow the exact solutions very well since the order of accuracy within a time step is 3 only. As a result, it cannot capture the fourth order load accurately. By using a seventh order accurate algorithm, close agreement is obtained. Alternatively, good agreement is also obtained by using smaller time steps for the sixth order accurate algorithm. The results from the sixth order accurate algorithm with 2 time steps almost coincide with the exact solutions within the time step. In Figures 6(a) and (b), the locations with better accuracy within the time step are also indicated.

5.6. FREE VIBRATION RESPONSES

After the loading duration T_s, the system is set into free vibration with initial conditions given by the responses at the end of T_s. The higher order accurate algorithms have three advantages over the Newmark method. First, with more accurate results obtained at the end of T_s, more accurate solutions are obtained for the free vibration responses. Second, a larger time step can be used to evaluate the free vibration. Third, the interpolation within each time step can be carried out to trace the free vibration responses.

For comparison, in the present study, the initial conditions for the free vibration evaluation for both the complex time step algorithms and the Newmark method are taken from the analytical results. The time step for the Newmark method is approximately one-tenth of the natural period (0·7 s) of the system and the results are not very satisfactory as shown in Figure 7. If the time step size is increased to approximately one-quarter of the natural period (1·75 s), the Newmark method fails to give meaningful results. Since the present sixth order accurate algorithm gives almost exact solutions at the end of the loading period, it is chosen to compute the free vibration responses as well. It can be seen from Figure 7 that using a time step of 1·75 s and with interpolation within each time step, the present sixth order accurate algorithm is able to trace the exact solutions closely for both displacement and velocity responses. The Newmark method will need to use a smaller time step (such as 0·35 s) in order to improve the solution's accuracy.

5.7. MULTI-DEGREE-OF-FREEDOM SYSTEMS

Consider a simply supported flexible panel subject to blast loading as described in reference [17]. The dimensions of the panel were 0·914 m by 1·219 m and 0·032 cm thick. The elastic modulus and Poisson's ratio were assumed to be $68{\cdot}95 \times 10^6$ kPa and 0·3, respectively. The panel was modelled as a simply supported beam, 0·914 m long and 25·4 mm wide with uniformly distributed blast loading on the beam. The beam was divided into 10 elements of equal length. The displacement time history at the mid-span is shown in Figure 8.

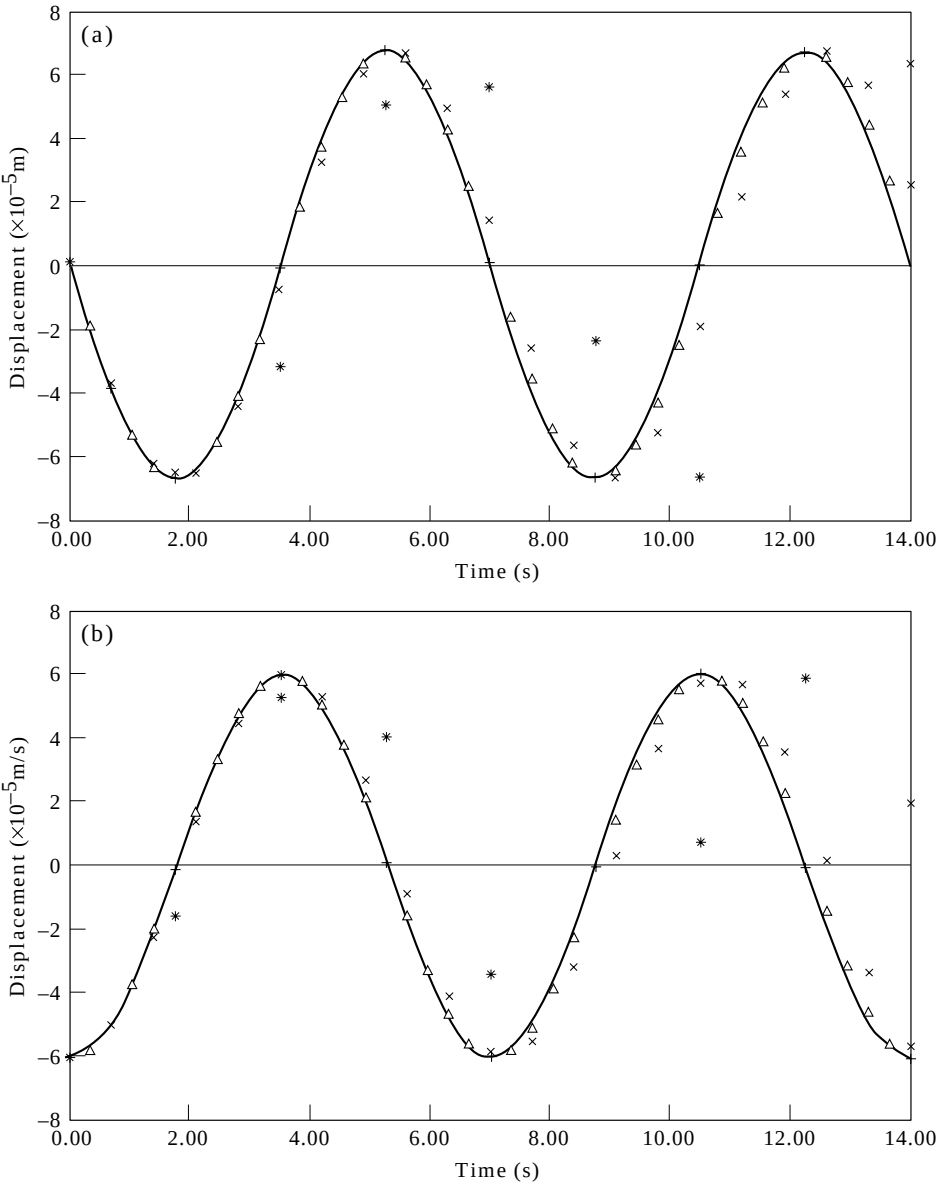

Figure 7. (a) Free vibration displacement. (b) Free vibration velocity. ———, Exact; △, Newmark ($\Delta t = 0{\cdot}35$ s); ×, Newmark ($\Delta t = 0{\cdot}7$ s); *, Newmark ($\Delta t = 1{\cdot}75$ s); +, 6th order ($\Delta t = 1{\cdot}75$ s).

From Figure 8, it can be seen that the complex time step seventh order algorithm can predict the blast loading responses accurately by using 3 time steps for the loading duration. The interpolated displacements almost coincide with the exact solutions. By applying the Newmark method, 20 time steps are insufficient to give comparable results. To achieve comparable results, approximately 300 time steps are required for the Newmark method. The computational efficiency of the present complex time step method is clearly seen.

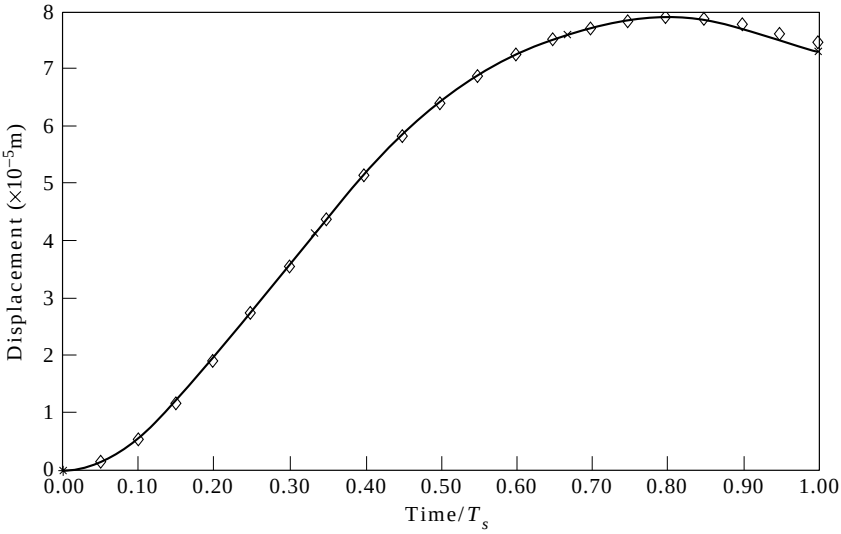

Figure 8. Mid-span displacement. ———, Exact ×, 7th order (3 time steps with interpolation); ◇, Newmark method (20 time steps).

6. CONCLUSIONS

In this paper, the complex time step method is extended to evaluate responses within a time step. The required weighting factors can be evaluated systematically. It is also found that there are some locations within a time step with one order higher in accuracy. To simplify the evaluation of excitation at the complex sub-step locations, a procedure is established to compute the values from the excitations sampled at discrete time locations within the time step.

The complex time step method is used to find the responses of a single-degree-of-freedom system subject to a fourth order blast loading. The present fifth, sixth and seventh order accurate algorithms are used to compute the results. It is found that the use of a single time step is sufficient for the sixth and higher order accurate algorithms to evaluate the responses at the end of the loading. Comparisons with the Newmark method show that the complex time step method is more attractive in evaluating responses due to high order loading.

The interpolation within a time step is also performed and the responses are accurate if the algorithm has fourth or higher order accuracy within the time step. As a result, seventh or higher order accurate algorithms are used to evaluate the responses with just one time step over the loading duration. In the free vibration phase, the present higher order algorithms are able to give accurate responses at the end as well as within each time step even when large time steps are used.

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